Complete List of Prime Number Rulers

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Abstract

Prime number ruler (PN ruler for short) of length L is a ruler where only prime numbers are used as its inside marks and it can measure any integer distances from 1 up to L as a difference between two marks (two ends are considered as marks). In this paper, we show the complete list of 28 minimal PN rulers by proving the necessary condition $L \leq 114$ for PN rulers and by searching all minimal PN rulers of $L \leq 114$ by a computer program.

1 Introduction

A *sparse ruler* is a ruler where some marks can be missing and it can measure any integer distances from 1 up to L as a difference between two marks (two ends are considered as marks here). Sparse rulers and their related works have been studied for a long time [4, 5, 8, 9], also appeared in some books [6] by Kobon Fujimura and Saburo Tamura, and [7] by Martin Gardner. There are some web sites maintaining the list of sparse rulers [2, 3].

Recently in 2013, an interesting ruler, named *prime number ruler* (PN ruler for short), is produced by the Institute of Fuben-eki System (organized by Hiroshi Kawakami of Kyoto University) [1]. It is a sparse ruler of length 18 where the first 7 prime numbers 2, 3, 5, 7, 11, 13, 17 are used as its inside marks, and it can measure any integer distances up to 18.

Now, you may have a question "Do there exist PN rulers other than length 18?" The answer is "Yes". There exist PN rulers using first n prime numbers when $n \leq 12$. For example, a ruler of length 38 with marks 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 can measure all distances up to 38. However, there are no PN rulers of any length using first 13 prime numbers.

Then, you may have another question "Do there exist infinite number of PN rulers?" The answer is "No".

In this paper, we prove there are only finite number of PN rulers by showing the necessary condition bouding the maximum length of PN rulers. We also show the the complete list of PN rulers found by our computer program.

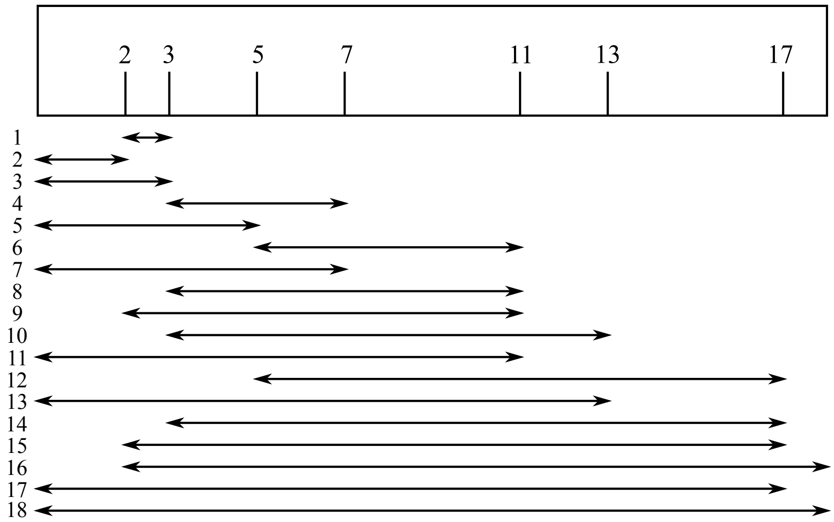

Figure 1: A PN ruler of length 18

2 Prime Number Rulers

Let M be a set specifying possible marks. A *ruler* of length L on M is a non-empty sequence $(a_1, a_2, \ldots, a_m)$ satisfying the following conditions where $a_0 = 0$ and $a_{m+1} = L$.

- $a_1, a_2, \ldots, a_m \in M$
- $0 = a_0 < a_1 < \cdots < a_m < a_{m+1} = L$

The sequence $(a_1, a_2, \ldots, a_m)$ is called the *marks* of the ruler. A ruler is called *complete* when any positive integer distance $d \leq L$ can be measured, that is, $d = a_j - a_i$ for some a_i and a_j $(0 \leq i < j \leq m + 1)$. A complete ruler is called *minimal* when the sequence is minimal, that is, any subsequence of its marks is not complete for the same length.

A *sparse ruler* of length L is a complete ruler of length L on $\mathbb{N}$ (a set of positive integers $\{1, 2, 3, \ldots\}$).

A *prime number ruler* (PN ruler for short) of length L is a complete ruler of length L on $\mathbb{P}$ (a set of prime numbers $\{p_1, p_2, p_3, \ldots\}$ where p_n is the n-th prime number).

Figure 1 displays a PN ruler $(2, 3, 5, 7, 11, 13, 17)$ of length 18. Each distance $d \leq 18$ can be measured as shown as arrows in the figure. The ruler is not minimal. Two minimal PN rulers of length 18 are $(2, 3, 5, 11, 13, 17)$ and $(2, 3, 5, 7, 11, 17)$.

3 Some Properties of Prime Number Rulers

Lemma 1. The length of a PN ruler is $p + 1$ for some prime number p.

Proof. Let L be the length of the PN ruler. To measure the distance $L - 1$, there should be a mark at 1 or $L - 1$. Therefore, $L - 1$ is a prime number. $\square$

By Lemma 1, the length L of a PN ruler is even when it is larger than 3.

Let $(a_1, a_2, \ldots, a_m)$ be a PN ruler of length L. We also assume $a_0 = 0$ and $a_{m+1} = L$. Since all prime numbers greater than 2 is odd, even numbers occurring in a_i's are only 0, 2, and L. Therefore, to measure an odd integer distance d, one of d, $d+2$, and $L-d$ should be a prime number.

With this observation, the following theorem is proved.

Proposition 1. The length a PN ruler can not exceed 114.

Proof. We assume there is a PN ruler of length $L > 114$, and derive a contradiction. The assumption $L > 114$ can be replaced by $L \geq 128$ since $L-1$ is a prime number from Lemma 1 and the next prime number larger than 113 is 127. Because any prime number larger than 3 is either $6m-1$ or $6m+1$ for some positive integer m, L is either $6m$ or $6m+2$. Now, we will derive contradictions for both cases.

- Case when $L = 6m \geq 128$. Let us consider to measure 33. As described in the above observation, either 33, 35, or $L-33$ should be a prime number. However, 33 and 35 are not prime, and $L - 33 = 6m - 33$ is divisible by 3. It contradicts the assumption.

- Case when $L = 6m + 2 \geq 128$. Let us consider to measure 119. Either 119, 121, or $L-119$ should be a prime number. However, 119 and 121 are not prime (divisible by 7 and 11 respectively), and $L - 119 = 6m - 117$ is divisible by 3. It contradicts the assumption.

Therefore, there are no PN rulers of length L when $L > 114$. $\square$

4 Complete List of Prime Number Rulers

In the previous section, $L \leq 114$ is shown as necessary condition for PN rulers.

We can list all possible length of PN rulers by checking whether $(p_1, p_2, \ldots, p_m)$ is a PN ruler of length $p_m + 1$ or not for each $m \in \{1, 2, \ldots, 30\}$ ($p_{30} = 113$). As the result, we obtain possible lengths of PN rulers as 3, 4, 6, 8, 12, 14, 18, 20, 24, 30, 32, 38, 44, and 62.

Table 1 shows the complete list of all minimal PN rulers found by our computer program. There are 28 minimal PN rulers. List of all PN rulers can be easily obtained by adding extra prime numbers. For example, $(2, 3, 5, 7, 11, 13)$ is another PN ruler of length 14. There are 102 PN rulers in total.

5 Conclusion

In this paper, we studied PN rulers where only prime numbers are used as its inside marks and it can measure any integer distances. We show there are only finite number of PN rulers by proving the necessary condition bounding the length of the PN ruler to 114. Through the execution of a computer program, we obtained the complete list of 28 minimal PN rulers.

Table 1: Complete List of 28 Minimal Prime Number Rulers

Length	#Marks	Marks
3	1	2
4	2	2, 3
6	2	2, 5
8	3	2, 3, 7
12	5	2, 3, 5, 7, 11
14	5	2, 3, 5, 7, 13
14	5	2, 3, 7, 11, 13
18	6	2, 3, 5, 7, 11, 17
18	6	2, 3, 5, 11, 13, 17
20	6	2, 3, 7, 11, 17, 19
20	7	2, 3, 5, 7, 11, 13, 19
20	7	2, 3, 5, 11, 13, 17, 19
24	7	2, 3, 5, 7, 11, 17, 23
24	8	2, 3, 5, 11, 13, 17, 19, 23
30	8	2, 3, 5, 7, 11, 17, 23, 29
32	8	2, 3, 7, 13, 17, 23, 29, 31
32	9	2, 3, 7, 11, 17, 19, 23, 29, 31
38	10	2, 3, 5, 7, 13, 19, 23, 29, 31, 37
44	11	2, 3, 5, 7, 11, 19, 23, 29, 31, 37, 43
44	11	2, 3, 5, 11, 13, 19, 23, 29, 37, 41, 43
44	11	2, 3, 5, 11, 17, 19, 23, 31, 37, 41, 43
44	11	2, 3, 5, 11, 19, 23, 29, 31, 37, 41, 43
44	11	2, 3, 7, 11, 13, 19, 23, 29, 37, 41, 43
44	11	2, 3, 7, 11, 17, 19, 23, 31, 37, 41, 43
44	11	2, 3, 7, 11, 19, 23, 29, 31, 37, 41, 43
62	14	2, 3, 7, 13, 17, 23, 29, 31, 37, 43, 47, 53, 59, 61
62	14	2, 3, 7, 13, 19, 23, 29, 31, 37, 41, 47, 53, 59, 61
62	14	2, 3, 7, 13, 19, 23, 29, 31, 37, 43, 47, 53, 59, 61

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